On the affine umbilical hypersurfaces

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In this paper we shall give an affine generalization of a theorem on the umbilical hypersurfaces of a Euclidean space ([1], p. 30).

Let A^{n+1} be the affine space of dimension $n+1$ and denote by R^{n+1} its coordinate space. Consider a smooth manifold M of dimension n, and an immersion $f: M \to A^{n+1}$ of M into A^{n+1}. Since our discussion is local, we may assume that M is a hypersurface imbedded in A^{n+1}.

Let be given an affine normalization $\xi: M \to R^{n+1}$ of the hypersurface M in A^{n+1}, that is a vector field ξ_x on M, the value of which is linearly independent from the tangent vectors of M at every point $x \in M$.

We denote the canonical covariant differentiation in A^{n+1} by ∇'. Let X and Y be tangent vector fields on M. Since $(\nabla'_X Y)_x$ is defined for each $x \in M$, we shall denote its tangential component with respect to the normalization ξ_x by $(\nabla_X Y)_x$, so we have

$$(\nabla'_X Y)_x = (\nabla_X Y)_x + g_x(X, Y)\xi_x. \tag{1}$$

It is known that $\nabla_X Y$ defines a symmetric linear connection on M, called the induced connection of M, and $g_x(X, Y)$ is a "scalar product" on M.

We say that the affine normalization $\xi: M \to R^{n+1}$ is relative, if the covariant derivative $(\nabla_X \xi)_x$ of ξ for each tangent vector $X \in TM$ has only tangential component, that is

$$(\nabla'_X \xi)_x = B_x(X), \tag{2}$$

where $B_x: T_x M \to T_x M$ is a linear operator of the tangent space of M.

We say that the normalized hypersurface (M, f, ξ) is affine umbilical at a point $x \in M$, if $B_x = \lambda_x I_x$, where λ_x is a scalar and I_x denotes the identity operator of the tangent space $T_x M$.

Lemma. *We have* $(\nabla_X B)(Y) = (\nabla_Y B)(X)$ *for any tangent vector field* X *and* Y *of* M.

Proof. Applying the equations (1) and (2) we get

$$B([X,Y]) = \nabla'_{[X,Y]}\xi = (\nabla'_X\nabla'_Y - \nabla'_Y\nabla'_X)\xi = \nabla'_X(B(Y)) - \nabla'_Y(B(X)) =$$
$$= \nabla_X(B(Y)) + g(X, B(Y))\xi - \nabla_Y(B(X)) - g(Y, B(X))\xi =$$
$$= (\nabla_X B)(Y) + B(\nabla_X Y) + g(X, B(Y))\xi - (\nabla_Y B)(X) - B(\nabla_Y X) - g(Y, B(X))\xi.$$

Since $\nabla_X Y$ is a symmetric covariant differentiation, we have $\nabla_X Y - \nabla_Y X = [X, Y]$, that is

$$B(\nabla_X Y) - B(\nabla_Y X) = B([X, Y]).$$

From this and the preceding calculation follows

$$\{(\nabla_X B)(Y) - (\nabla_Y B)(X)\} + \{g(X, B(Y)) - g(Y, B(X))\}\cdot\xi = 0,$$

which shows that the tangential component is $(\nabla_X B)(Y) - (\nabla_Y B(X) = 0$.

Thus the lemma is proved.

We say that the affine normalization $\xi\colon M \to R^{n+1}$ of the hypersurface (M, f) is radial affine, if there is a coordinate system of A^{n+1} in which the affine normal vector ξ_x and the position vector of the point $f_x \in A^{n+1}$ coincide for each point $x \in M$.

It is trivial that every radial affine normalization is relative affine.

We say that the affine normalization $\xi\colon M \to R^{n+1}$ is similar to a radial affine normalization, if there exists a nonzero constant τ such that $\tau\cdot\xi\colon M \to R^{n+1}$ is a radial affine normalization.

Theorem 1. *Let (M, f, ξ) be an affine normalized hypersurface in A^{n+1}. If the affine normalization $\xi\colon M \to R^{n+1}$ is similar to a radial affine normalization, then the hypersurface is affine umbilical at every point $x \in M$.*

Proof. Identifying $f_x \in A^{n+1}$ with the corresponding position vector in R^{n+1}, we have $\xi_x = 1/\tau\, f_x$, which shows that for every tangent vector field X on M $(\nabla'_X\xi)_x = 1/\tau\, X_x$.

It follows that $B_x = 1/\tau\, I_x$, that is the hypersurface is affine umbilical.

Theorem 2. *Let (M, f, ξ) be a relative affine normalized hypersurface in A^{n+1}. If every point $x \in M$ is affine umbilical, then either the vector function $\xi\colon M \to R^{n+1}$ is constant or the normalization ξ is similar to a radial affine normalization of the hypersurface $f\colon M \to A^{n+1}$.*

Proof. Since every point $x \in M$ is affine umbilical, there exists a scalar function λ_x on M such that $B_x = \lambda_x I_x$. We are going to prove that λ_x is a constant function.

For any tangent vector fields X and Y on M we have

$$(\nabla_X B)(Y) = \nabla_X(BY) - B(\nabla_X Y) = \nabla_X(\lambda Y) - \lambda\cdot\nabla_X Y = (X\lambda)Y.$$

Similarly,

$$(\nabla_Y B)(X) = (Y\lambda)X.$$

By Lemma we obtain

$$(X\lambda)Y = (Y\lambda)X.$$

For each $x \in M$ we may choose the tangent vector fields X and Y on M so that X_x and Y_x are linearly independent. It follows that $Z\lambda = 0$ for every $Z \in T_x M$, that is λ is equal to a constant on M.

Identifying $f_x \in A^{n+1}$ with its position vector in R^{n+1}, we consider $f_x + \xi_x$ as an R^{n+1}-valued vector function on M.

If X is an element of $T_x M$, we have

$$\nabla'_X(\lambda f - \xi) = \lambda X - \nabla'_X \xi = \lambda X - \lambda X = 0,$$

which shows that $\lambda f_x - \xi_x = \alpha$ is a constant vector in R^{n+1}.

If $\lambda = 0$, then ξ_x is a constant vector, and our assertion is proved.

If $\lambda \neq 0$, then $\lambda f_x - \xi_x = \alpha$ implies $f_x - 1/\lambda\, \alpha = 1/\lambda \cdot \xi_x$, which shows that in the coordinate system of A^{n+1} translated with the vector $1/\lambda \cdot \alpha$ the affine normalization $1/\lambda\, \xi$ is a radial affine normalization of the hypersurface (M, f). Thus our theorem is proved.

Reference

[1] S. Kobayashi, K. Nomizu, *Foundations of differential geometry.* Vol. II, Interscience Publishers (New York—London—Sydney, 1969).

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